Counting

Professor Don Colton

Brigham Young University Hawaii

1 What Are We Trying To Do?

It can be important to know approximately or exactly how many of a thing can exist, or how many ways a thing can be done.

In this paper, we briefly look at six of those ways and show how to calculate the count for each.

2 Distinct Assignments

Say we have N houses and K colors of paint. Each house must be painted with one of those colors. (We keep it simple; there is no mixing of paint.)

How many ways can this be done?

The answer is that each house can be painted K different ways. Those ways do not influence each other in any way.

So, we have K ways for the first house, and K ways for the second house, and K ways for the third house, and so on.

We have K times K ..., for N Ks, or K^N, K to the Nth power.

This relies on the multiplication rule, where the total number of possibilities is calculated by multiplying the number of possibilities of each individual piece.

3 Permutation

Given N distinct (individual) objects, in how many different ways can they be listed (or arranged)?

The answer is $N!$, N factorial.

There are N choices for the item to be listed first. Then there are N-1 items from which we can choose the item to be listed second. Eventually, there is only one item that can be listed last.

$N!$ means $N \times (N-1) \times (N-2) \times (N-3) \times ... \times 3 \times 2 \times 1$.

The recursive definition is $N! = N \times (N-1)!$, and the basis case is $1! = 1$. (It is also generally agreed that $0! = 1$.)

Permutation is also called Ordered Full Set Without Replacement.

4 Ordered Selection

Short of a full permutation, we can stop after selecting K of the items. We then have an ordered selection of K items out of a total set of N items.

Just like with permutation, we calculate as follows:

There are N choices for the item to be listed first. Then there are N-1 items from which we can choose the item to be listed second. Eventually, there are N-K choices for the item that can be listed last.

$N \times (N-1) \times (N-2) \times (N-3) \times ... \times (N-K+1)$ which equals $N!/(N-K)!$.

If we want to list three items out of a set of seven possible items, the answer would be:

$7! \ / \ 4! = 7 \times 6 \times 5 = 210$

5 Unordered Selection

With an unordered selection, we don't care the order in which the items are selected. We can simply make an ordered selection, and then cancel out the different orderings in which the same items might appear.

This is also called "choose", as in "7 choose 3".

If we want an unordered group of three items out of a set of seven possible items, the answer would be:

$7! \ / \ 4! = 7 \times 6 \times 5 = 210$ for the ordered lists,

$210 \ / \ 3 \ / \ 2 \ / \ 1 = 35$ after noticing that there are $3 \times 2 \times 1$ ways in which each of the selections can appear.

$7! \ / \ 4! \ / \ 3! = 7 \times 6 \times 5 \ / \ 3 \ / \ 2 \ / \ 1 = 7 \times 5 = 35$

The formula is $N!/(N-K)!/K!$

6 Orderings With Identical Items

In the above examples, the items were always distinctive. They could always be told apart from one another.

When we have identical items, we assume the items cannot be told apart. Another way to say this is to call them "unlabeled" items.

A classic example is ordering letters to form "words."

In how many distinct ways can the letters *aaabbc* be arranged?

Well, if the letters were distinct, we would have 6!. But since they are not distinct, we have to divide by the amount of duplication that exists.

Since there are three "a"s in the string, we divide by 6. The "a"s could have been arranged as $a_1a_2a_3$ or $a_1a_3a_2$ or four other ways.

Since there are two "b"s in the string, we divide by 2. The "b"s could have been arranged as b_1b_2 or b_2b_1.

Since there is only one "c" we divide by 1 (i.e., make no adjustment). The "c"s could have been arranged as c_1 only.

The answer will then be:

$6 \times 5 \times 4 \times 3 \times 2 \times 1 / 3 / 2 / 1 / 2 / 1 / 1 = 60$

That is, $6!/3!/2!/1! = 60$

7 Distributing Objects Into Bins

In this case, we have N bins. Each bin is distinct and identifiable. Each could represent a child that is receiving gifts.

The objects may include duplicates. For example, we may want to distribute 2 baseballs among 3 children. How many ways can this be done?

There are six ways, listed as follows, with each digit telling how many baseballs a certain child got. (ABC means A for the first child, B for the second child, and C for the third child.)

200, 110, 101, 020, 011, 002

Nobody said it had to be fair.

The calculation for this is just about like Orderings With Identical Items. But not quite.

We introduce a fake item, the boundary line between bins (children). We will use X to represent the boundary, an b to represent the baseball.

200 can be written as bbXX.

110 can be written as bXbX.

002 can be written as XXbb.

In short, we can restate the problem as how many different words can be constructed from the letters "XXbb".

That would be $4!/2!/2! = 4 \times 3 \times 2 \times 1 / 2 / 1 / 2 / 1 = 4 \times 3 / 2 = 6$.

What if there are more items than just baseballs?

We use the multiplication rule, handling each of the items separately.

Say we have three children, two baseballs, and two bats.

We have 6 ways to divide the baseballs. We also have 6 ways to divide the bats. The total is 36.

Say we have 4 kids, 3 balls, 2 gloves, and 1 bat.

For the balls, we have XXXbbb $= 6!/3!/3! = 20$.

For the gloves, we have XXXgg $= 5!/3!/2! = 10$.

For the bat, we have XXXb $= 4!/3!/1! = 4$. Well, that's kind of obvious when you think about it. One bat, four kids? Four ways.

Now we multiply those together, since the assignments are independent:

$20 \times 10 \times 4 = 800$

There are 800 ways to distribute 3 balls, 2 gloves, and 1 bat among 4 kids.